D-modules on affine toric varieties and A-hypergeometric systems

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Abstract.

Let A be an integer matrix of size (d, n) with $\mathrm{rank}(A) = d$, and let $D(R_A)$ be the ring of differential operators on the affine toric variety defined by A.

We show that the classification of A-hypergeometric systems and that of ZA-graded simple $D(R_A)$-modules are the same.

We also give conditions for the algebra $D(R_A)$ being simple.

- **Differential operators.**

Let

$$R : \text{a } C\text{-algebra, commutative,}$$
$$M, N : \text{(left) } R\text{-modules.}$$

Define inductively

$$D(M,N) := \bigcup_{k=0}^{\infty} D^k(M,N) \subseteq \mathrm{Hom}_C(M,N)$$

by

$$D^0(M,N) := \mathrm{Hom}_R(M,N),$$
$$D^{k+1}(M,N) := \{f \in \mathrm{Hom}_C(M,N) : rf - fr \in D^k(M,N) \ (\forall r \in R)\}.$$

Then

$$D(M) := D(M,M) \subseteq \mathrm{End}_C(M)$$

is a C-algebra, and $D(M,N)$ is a $(D(N), D(M))$-bimodule.

• Affine semigroup algebra.

Let

$$A = (a_1, a_2, \dots, a_n) \in M_{d\times n}(Z).$$

$$NA := Na_1 + Na_2 + \cdots Na_n.$$

Assume $ZA = Z^d$.

$$R_A := C[NA] = \bigoplus_{a\in NA} Ct^a$$
$$\subseteq C[t_1^{\pm 1}, \dots t_d^{\pm 1}] = C[ZA].$$

Then $\quad R_A = R/I_A, \quad$ where

$$R := C[x] := C[x_1, \dots, x_n],$$

$$I_A :=$$
$$\langle x^u - x^v : Au = Av \ (u, v \in N^n)\rangle.$$

- $D(C[t_1^{\pm 1}, \ldots t_d^{\pm 1}])$.

$$
\begin{aligned}
&C\langle t^{\pm 1}, \partial_t\rangle \\
&\quad := C[t_1^{\pm 1}, \ldots t_d^{\pm 1}]\langle \partial_{t_1}, \ldots, \partial_{t_d}\rangle \\
&\quad = D(C[t_1^{\pm 1}, \ldots t_d^{\pm 1}]).
\end{aligned}
$$

Weight decomposition:

$$C\langle t^{\pm 1}, \partial_t\rangle = \bigoplus_{b \in Z^d} C\langle t^{\pm 1}, \partial_t\rangle_b,$$

$$
\begin{aligned}
&C\langle t^{\pm 1}, \partial_t\rangle_b \\
&\quad = \{P \in C\langle t^{\pm 1}, \partial_t\rangle : [s_i, P] = b_i P \ (\forall i)\} \\
&\quad = t^b C[s].
\end{aligned}
$$

Here

$$s_i := t_i \partial_{t_i}, \quad C[s] := C[s_1, \ldots, s_d].$$

• $D(R_A)$.

$$D(R_A) \\ = \{P \in C\langle t^{\pm 1}, \partial_t\rangle : P(R_A) \subseteq R_A\}.$$

Weight decomposition:

$$D(R_A) = \bigoplus_{b \in Z^d} D(R_A)_b,$$

where

$$D(R_A)_b = D(R_A) \cap C\langle t^{\pm 1}, \partial_t\rangle_b.$$

See $s_1, \ldots, s_d \in D(R_A)$.

($s_i.t^b = b_i t^b$. Hence $s_i(R_A) \subseteq R_A$.)

Hence

$$D(R_A)_0 = C[s] = C\langle t^{\pm 1}, \partial_t\rangle_0.$$

• **Category $\mathcal{O}$.**

$\mathcal{O}$ is a full subcategory of the category of left $D(R_A)$-modules.

$M \in \mathcal{O}$

$$\overset{\text{def.}}{\Leftrightarrow} \; M = \bigoplus_{\alpha \in C^d} M_\alpha,$$

$$M_\alpha = \{x \in M : s_i.x = \alpha_i x \ (\forall i)\}.$$

Each M_α is finite-dimensional.

For $\alpha \in C^d$, define

$$\begin{aligned} M(\alpha) &:= D(R_A)/D(R_A)\langle s - \alpha \rangle \\ &:= D(R_A)/\sum_{i=1}^{d} D(R_A)\langle s_i - \alpha_i \rangle. \end{aligned}$$

Then

$M(\alpha) = \bigoplus_{\lambda \in \alpha + Z^d} M(\alpha)_\lambda$,
$M(\alpha) \in \mathcal{O}$.

Proposition ([Musson-Van den Bergh]).

(1) $\mathrm{Hom}_{D(R_A)}(M(\alpha), M) = M_\alpha$ $(\forall M \in \mathcal{O})$.

(2) $M(\alpha)$ is a projective object in $\mathcal{O}$.

(3) $M(\alpha)$ has a unique simple quotient (denoted by $L(\alpha)$).

(4) All simple objects in $\mathcal{O}$ are of the form $L(\alpha)$.

(5) $M(\alpha) \longrightarrow L(\alpha)$ is the projective cover.

(6) $M(\alpha) \simeq M(\beta) \Leftrightarrow L(\alpha) \simeq L(\beta)$.

Proof. (1)

$$\mathrm{Hom}_{D(R_A)}(M(\alpha), M) \ni f \mapsto f(\bar{1}) \in M_\alpha.$$

(2) $\mathrm{Hom}_{D(R_A)}(M(\alpha), \bullet)$ is exact by (1).

(3) Since $M(\alpha)_\alpha = C[s]/(s-\alpha)$,

we have $\dim_C M(\alpha)_\alpha = 1$.

Hence a submod. $L \subseteq M(\alpha)$ is proper iff $\Leftrightarrow L_\alpha = 0$.

Hence $\sum_{L \subseteq M(\alpha),\, L_\alpha = 0} L$ is the biggest submod. of $M(\alpha)$. (5).

$$L(\alpha) = M(\alpha) / \sum_{L \subseteq M(\alpha),\, L_\alpha = 0} L.$$

(4) Let L be simple in $\mathcal{O}$.

Then $\exists \alpha$ s.t. $L_\alpha \neq 0$.

Then $\exists f : M(\alpha) \to L$ by (1).

Since L is simple, f is surjective.

Hence $L = L(\alpha)$ by def.

(3) and (5) imply (6).

• **Right modules.**

${}^R\mathcal{O}$ **is a full subcategory of the category of right** $D(R_A)$**-modules.**

$M \in {}^R\mathcal{O}$
$\overset{\text{def.}}{\Leftrightarrow}$ $M = \bigoplus_{\alpha \in C^d} M_\alpha,$

$M_\alpha = \{x \in M : x.s_i = -\alpha_i x \ (\forall i)\}.$

Each M_α **is finite-dimensional.**

For $\alpha \in C^d$**, define**

$${}^R M(\alpha) := D(R_A)/\langle s - \alpha \rangle D(R_A).$$

Then

${}^R M(\alpha) = \bigoplus_{\lambda \in -\alpha + Z^d} {}^R M(\alpha)_\lambda,$
${}^R M(\alpha) \in {}^R\mathcal{O}.$

Proposition (^{R}[Musson-Van den Bergh]).

(1) $\mathrm{Hom}_{D(R_A)}({}^R M(\alpha), M) = M_{-\alpha}$
$(\forall M \in {}^R\mathcal{O})$.

(2) ${}^R M(\alpha)$ **is a proj. object in** ${}^R\mathcal{O}$.

(3) ${}^R M(\alpha)$ **has a unique simple quotient (denoted by** ${}^R L(\alpha)$**).**

(4) All simple objects in ${}^R\mathcal{O}$ **are of the form** ${}^R L(\alpha)$.

(5) ${}^R M(\alpha) \to {}^R L(\alpha)$ **is the projective cover.**

(6) ${}^R M(\alpha) \simeq {}^R M(\beta)$
$\Leftrightarrow {}^R L(\alpha) \simeq {}^R L(\beta)$.

For $M \in \mathcal{O}$ $({}^R\mathcal{O})$, set

$$M^* := \bigoplus_{\lambda} M^*_{\lambda},$$

where $M^*_{\lambda} = \mathrm{Hom}_C(M_{-\lambda}, C)$.

Then $M^* \in {}^R\mathcal{O}$ $(\mathcal{O})$, $*$ is a duality functor.

Proposition.

(1) $\mathrm{Hom}_{D(R_A)}(M, {}^R M(\alpha)^*)$
$= \mathrm{Hom}_C(M_\alpha, C)$.

(2) ${}^R M(\alpha)^*$ is an inj. object in $\mathcal{O}$.

(3) ${}^R M(\alpha)^*$ has the unique simple subobject ${}^R L(\alpha)^*$ in $\mathcal{O}$.

(4) $L(\alpha) \simeq {}^R L(\alpha)^*$.

(5) ${}^R L(\alpha)^* \to {}^R M(\alpha)^*$ is the inj. hull.

(6) $L(\alpha) \simeq L(\beta)$ iff ${}^R L(\alpha) \simeq {}^R L(\beta)$.

- **Functors.**

$$R = C[x_1, \dots, x_n] \to R_A = R/I_A.$$

$$D(R) = C\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$$
:the n-th Weyl algebra.

$$D(R, R_A) = D(R)/I_A D(R)$$
is a $(D(R_A), D(R))$-bimodule.

We define functors

$$\Phi : \mathrm{Mod}^R(D(R_A)) \to \mathrm{Mod}^R_{I_A}(D(R))$$

and

$$\Psi : \mathrm{Mod}^R_{I_A}(D(R)) \to \mathrm{Mod}^R(D(R_A))$$

by

$$\Phi(M) := M \otimes_{D(R_A)} D(R, R_A),$$

$$\Psi(N) := \mathrm{Hom}_{D(R)}(D(R, R_A), N).$$

Φ is the direct image functor, for right D-modules, of the closed inclusion

$$V(I_A) \to C^n.$$

We may think $D(R_A) \subseteq D(R, R_A)$:

$$t^{a_j} = x_j$$

$$s_i = \sum_{j=1}^{n} a_{ij}\theta_j.$$

Here

$$\theta_j = x_j\partial_j, \qquad a_j = {}^t(a_{1j}, \ldots, a_{dj}).$$

- **A-HGS.**

$$H(\alpha) := D(R)/D(R)I_A(\partial)+D(R)\langle A\theta-\alpha\rangle.$$

Here

$I_A(\partial)$: **the toric ideal in** $C[\partial_1, \ldots, \partial_n]$.

$$D(R)\langle A\theta - \alpha\rangle = \sum_{i=1}^{d} D(R) \sum_{j=1}^{n} (a_{ij}\theta_j - \alpha_i).$$

ι : **the anti-auto. of** $D(R)$ **interchanging** x_j **and** ∂_j $(\forall j)$.

ι **interchanges left/right** $D(R)$**-modules.**

$${}^{R}H(\alpha) := \iota(H(\alpha)) = D(R)/I_A D(R) + \langle A\theta - \alpha\rangle D(R).$$

Note $\iota(x_j\partial_j) = \iota(\partial_j)\iota(x_j) = x_j\partial_j$.

$$\Phi({}^R M(\alpha)) = M(\alpha) \otimes D(R, R_A)$$
$$= D(R_A)/\langle s - \alpha \rangle D(R_A) \otimes_{D(R_A)} D(R)/I_A D(R)$$
$$= {}^R H(\alpha).$$

Theorem.

T.F.A.E.

(1) $M(\alpha) \simeq M(\beta)$.

(2) $L(\alpha) \simeq L(\beta)$.

(3) ${}^R L(\alpha) \simeq {}^R L(\beta)$.

(4) ${}^R M(\alpha) \simeq {}^R M(\beta)$.

(5) ${}^R H(\alpha) \simeq {}^R H(\beta)$.

(6) $H(\alpha) \simeq H(\beta)$.

(7) $\alpha \sim \beta$.

- **Sketch of (5) $\Rightarrow$ (4).**

First show

$$\mathrm{End}_{D(R)}({}^{R}H(\alpha)) = C id.$$

So

$$\dim \mathrm{Hom}_{D(R)}({}^{R}H(\alpha), {}^{R}H(\beta)) = 1.$$

Then

$$D(R)_{\beta-\alpha} \twoheadrightarrow \mathrm{Hom}_{D(R)}({}^{R}H(\alpha), {}^{R}H(\beta)).$$

Hence

$$\mathrm{Hom}_{D(R)}({}^{R}H(\alpha), {}^{R}H(\beta)) \simeq \mathrm{Hom}_{D(R)}({}^{R}M(\alpha), {}^{R}M(\beta)).$$

• **Primitive ideals.**

Theorem.

$\{\mathrm{Ann}\, L(\alpha) : \alpha \in C^d\}$ is finite.

Proposition.

$D(R_A)$ is simple

$$\Leftrightarrow \quad \mathrm{Ann}\, L(\alpha) = 0 \quad (\forall \alpha).$$

Proposition.

$\mathrm{Ann}\, L(\alpha) = 0 \Leftrightarrow \mathrm{ZC}([\alpha]) = C^d$

([M-VdB] under the S_2 condition).

Here $[\alpha]$ is the equivalence class of α w.r.t. $\sim$.

- $R_{>0}(\alpha)$, **extreme classes.**

Set

$$\mathcal{F}_+(\alpha) := \{\sigma : \text{facet}, F_\sigma(\alpha) \in F_\sigma(NA)\},$$

$$\mathcal{F}_-(\alpha) := \{\sigma : \text{facet}, F_\sigma(\alpha) \in Z \setminus F_\sigma(NA)\},$$

and

$$R_{>0}(\alpha) := \{\gamma \in R^d : F_\sigma(\gamma) > 0\,(\forall \sigma \in \mathcal{F}_+(\alpha)) \quad F_\sigma(\gamma) < 0\,(\forall \sigma \in \mathcal{F}_-(\alpha))\}.$$

α: **extreme**

$\overset{\text{def.}}{\Leftrightarrow}$ **For every** τ, $E_\tau(\alpha)$ **is** $\emptyset$ **or has as many elements as possible.**
(i.e., $[Q(A\cap\tau)\cap Z^d : Z(A\cap\tau)]$**-many.)**

Example.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

$$F_{\sigma_{14}}(s) = s_2, \quad F_{\sigma_{36}}(s) = 3s_1 - s_2,$$

$$F_{\sigma_{123}}(s) = s_3, \quad F_{\sigma_{456}}(s) = s_1 - s_3,$$

$$NA = \{a \in R_{\geq 0}A \cap Z^3 : F_{\sigma_{14}}(a) \neq 1\}.$$

Let $\alpha := {}^t(0,1,0)$. **Then**

$$\mathcal{F}_+ = \{\sigma_{123}, \sigma_{456}\}, \quad \mathcal{F}_- = \{\sigma_{14}, \sigma_{36}\}.$$

Since $F_{\sigma_{14}} + F_{\sigma_{36}} = 3(F_{\sigma_{123}} + F_{\sigma_{456}})$,

$R_{>0}(\alpha) = \emptyset$.

Example.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} = [a_1, a_2, a_3].$$

Then

$$[Q(A \cap \sigma_3) \cap Z^2 : Z(A \cap \sigma_3)] = 2,$$

where $\sigma_3 = R_{\geq 0} a_3$.

$$E_{\sigma_3}(0) = \{0\} \not\ni {}^t(1, 0).$$

Hence 0 is not extreme.

Theorem.

$$\begin{aligned} &\alpha \text{ is extreme and } R_{>0}(\alpha) \neq \emptyset \\ \Leftrightarrow\ &\mathrm{ZC}([\alpha]) = C^d \\ \Leftrightarrow\ &\mathrm{Ann}\, L(\alpha) = 0. \end{aligned}$$

Theorem.

$$D(R_A)\text{: simple} \Leftrightarrow \begin{cases} NA\text{: scored} \\ R_{>0}(\alpha) \neq \emptyset\,(\forall \alpha). \end{cases}$$

Remark.

Need to check $R_{>0}(\alpha) \neq \emptyset$ for finitely many α, since there exist only finitely many primitive ideals.

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