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Introduction to A-hypergeometric systems

Using Gauss's hypergeometric equation, I explain how its corresponding A-hypergeometric system is constructed. This is mainly based on the papers:

Miller: Lie theory and generalizations of hypergeometric functions, SIAM J. Appl. Math. 25 (1973) 226–235.

Hrabowski: Multiple hypergeometric functions and simple Lie algebras SL and Sp, SIAM J. Math. Anal. 16 (1985) 876–886.

Some fundamental results are also given.

- **Gauss's hypergeometric function**

$$F(\alpha,\beta,\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!},$$

where $\gamma \notin Z_{\leq 0}$,

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}=a(a+1)\cdots(a+k-1).$$

• Obvious Contiguity Relations

$$(1)(\theta_z + \alpha)F(\alpha,\beta,\gamma;z) = \alpha F(\alpha+1,\beta,\gamma;z),$$

$$(2)(\theta_z + \beta)F(\alpha,\beta,\gamma;z) = \beta F(\alpha,\beta+1,\gamma;z),$$

$$(3)(\theta_z + \gamma - 1)F(\alpha,\beta,\gamma;z) = (\gamma-1)F(\alpha,\beta,\gamma-1;z),$$

$$(4)\frac{d}{dz}F(\alpha,\beta,\gamma;z) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;z).$$

where $\theta_z = z\frac{d}{dz}$.

Proof.

$$(\theta_z+\alpha)((\alpha)_k z^k) = (k+\alpha)(\alpha)_k z^k$$
$$= \alpha(\alpha+1)_k z^k.$$

- **Shift of Parameter Vector**

$$(1)\quad a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (2)\quad a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$(3)\quad a_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad (4)\quad a_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

See $a_1 + a_2 = a_3 + a_4$.

- **Gauss's Hypergeometric Equation**

$$\begin{aligned} &(\theta_z + \gamma)\frac{d}{dz}F(\alpha, \beta, \gamma; z) \\ &\qquad = \alpha\beta F(\alpha + 1, \beta + 1, \gamma; z), \\ &(\theta_z + \alpha)(\theta_z + \beta)F(\alpha, \beta, \gamma; z) \\ &\qquad = \alpha\beta F(\alpha + 1, \beta + 1, \gamma; z). \end{aligned}$$

So $F(\alpha, \beta, \gamma; z)$ satisfies

$$[(\theta_z + \gamma)\frac{d}{dz} - (\theta_z + \alpha)(\theta_z + \beta)]\phi = 0.$$

- **Make the contiguity operators parameter-free vector fields.**

Introduce new variables $u_\alpha, u_\beta, u_\gamma$ corresponding to the parameter α, β, γ.

Consider

$$\tilde{F}(\alpha, \beta, \gamma; u_\alpha, u_\beta, u_\gamma, z) = u_\alpha^\alpha u_\beta^\beta u_\gamma^{\gamma-1} F(\alpha, \beta, \gamma; z)$$

Then

$$\begin{aligned} & u_\alpha(\theta_z + \theta_\alpha)\tilde{F}(\alpha, \beta, \gamma; u_\alpha, u_\beta, u_\gamma, z) \\ = {} & u_\alpha(\theta_z + \theta_\alpha) u_\alpha^\alpha u_\beta^\beta u_\gamma^{\gamma-1} F(\alpha, \beta, \gamma; z) \\ = {} & u_\alpha u_\alpha^\alpha u_\beta^\beta u_\gamma^{\gamma-1} (\theta_z + \alpha) F(\alpha, \beta, \gamma; z) \\ = {} & u_\alpha^{\alpha+1} u_\beta^\beta u_\gamma^{\gamma-1} \alpha F(\alpha + 1, \beta, \gamma; z) \\ = {} & \alpha \tilde{F}(\alpha + 1, \beta, \gamma; u_\alpha, u_\beta, u_\gamma, z). \end{aligned}$$

- **Contiguity Relations for $\tilde{F}$**

$$
\begin{aligned}
&(5)\ u_\alpha(\theta_z+\theta_\alpha)\tilde{F} = \alpha\tilde{F}(\alpha+1),\\
&(6)\ u_\beta(\theta_z+\theta_\beta)\tilde{F} = \beta\tilde{F}(\beta+1),\\
&(7)\ u_\gamma^{-1}(\theta_z+\theta_\gamma)\tilde{F} = (\gamma-1)\tilde{F}(\gamma-1),\\
&(8)\ u_\alpha u_\beta u_\gamma \frac{\partial}{\partial z}\tilde{F}\\
&\qquad = \frac{\alpha\beta}{\gamma}\tilde{F}(\alpha+1,\beta+1,\gamma+1).
\end{aligned}
$$

These four operators commute with one another.

$$
\begin{aligned}
&u_\gamma^{-1}(\theta_z+\theta_\gamma)\cdot u_\alpha u_\beta u_\gamma \frac{\partial}{\partial z}\tilde{F}\\
&\qquad\qquad = \alpha\beta\tilde{F}(\alpha+1,\beta+1),\\
&u_\alpha(\theta_z+\theta_\alpha)\cdot u_\beta(\theta_z+\theta_\beta)\tilde{F}\\
&\qquad\qquad = \alpha\beta\tilde{F}(\alpha+1,\beta+1).
\end{aligned}
$$

Hence

$$[u_\gamma^{-1}(\theta_z+\theta_\gamma)\cdot u_\alpha u_\beta u_\gamma \frac{\partial}{\partial z} \\ -u_\alpha(\theta_z+\theta_\alpha)\cdot u_\beta(\theta_z+\theta_\beta)]\tilde{F}=0.$$

Clearly $\tilde{F}(\alpha,\beta,\gamma;u_\alpha u_\beta,u_\gamma,z)$ also satisfies

$$\theta_\alpha\tilde{F}=\alpha\tilde{F}, \tag{9}$$

$$\theta_\beta\tilde{F}=\beta\tilde{F}, \tag{10}$$

$$\theta_\gamma\tilde{F}=(\gamma-1)\tilde{F}. \tag{11}$$

- **Change of Variables**

$$(u_\alpha, u_\beta, u_\gamma, z) \to (x_1, x_2, x_3, x_4)$$

so that the contiguity operators in (5)-(8) are transformed into

$$\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial x_4}.$$

Explicitly put

$x_1 = -u_\alpha^{-1}, \quad x_2 = -u_\beta^{-1}, \quad x_3 = u_\gamma$

$x_4 = u_\alpha^{-1} u_\beta^{-1} u_\gamma^{-1} z.$

Then

$$\frac{\partial}{\partial x_1} = u_\alpha(\theta_z + \theta_\alpha), \quad \frac{\partial}{\partial x_2} = u_\beta(\theta_z + \theta_\beta),$$
$$\frac{\partial}{\partial x_3} = u_\gamma^{-1}(\theta_z + \theta_\gamma), \quad \frac{\partial}{\partial x_4} = u_\alpha u_\beta u_\gamma \frac{\partial}{\partial z}.$$

and

$$\begin{aligned} \theta_\alpha &= -\theta_1 - \theta_4, \\ \theta_\beta &= -\theta_2 - \theta_4, \\ \theta_\gamma &= \theta_3 - \theta_4. \end{aligned}$$

Hence $\tilde{F}(\alpha,\beta,\gamma;x_1,x_2,x_3,x_4)$ satisfies

$$\begin{aligned}
&(\theta_1+\theta_4+\alpha)\Phi=0,\\
&(\theta_2+\theta_4+\beta)\Phi=0,\\
&(-\theta_3+\theta_4+\gamma-1)\Phi=0,\\
&(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_3}\frac{\partial}{\partial x_4})\Phi=0.
\end{aligned}$$

This is the A-hypergeometric system with parameter vector ${}^t(-\alpha,-\beta,1-\gamma)$, where

$$A=\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&-1&1\end{pmatrix}=(a_1,a_2,a_3,a_4).$$

$\frac{\partial}{\partial x_j}$ is a contiguity operator shifting parameter by $-a_j$.

• General Hypergeometric Series

Let $\alpha \in C^{s+t}$,

$$A = (a_1, a_2, \ldots, a_l) = (a_{ij}) \\ \in \mathrm{Mat}_{(s+t)\times l}(Z).$$

$$f(\alpha; z) \\ = \sum_{m \in N^l} \frac{\prod_{i=1}^{s} (\alpha_i)_{\sum_{j=1}^{l} a_{ij} m_j}}{\prod_{i=1}^{t} (\alpha_{s+i})_{\sum_{j=1}^{l} a_{s+i\,j} m_j}} \frac{z^m}{m!}.$$

• Obvious Contiguity Relations

For $1 \le p \le s$,

$$\alpha_p f(\alpha+1_p; z) = (\alpha_p + \sum_{j=1}^{l} a_{pj}\theta_{z_j}) f(\alpha; z).$$

For $s+1 \le p \le s+t$,

$$(\alpha_p - 1) f(\alpha - 1_p; z) = (\alpha_p - 1 + \sum_{j=1}^{l} a_{pj}\theta_{z_j}) f(\alpha; z).$$

For $1 \le p \le l$,

$$\partial_{z_p} f(\alpha; z) = \frac{\prod_{i=1}^{s} (\alpha_i)_{a_{ip}}}{\prod_{i=s+1}^{s+t} (\alpha_i)_{a_{ip}}} f(\alpha + a_p; z).$$

Introduce new variables $u_1, \dots, u_{s+t}$, and let

$$\begin{aligned}&\tilde{f}(\alpha; z, u)\\ &= f(\alpha; z)\left(\prod_{i=1}^{s} u_i^{\alpha_i}\right)\left(\prod_{i=s+1}^{s+t} u_i^{\alpha_i - 1}\right).\end{aligned}$$

Change the variables:

$$\begin{aligned} x_p &= -u_p^{-1} \qquad (1 \le p \le s)\\ x_p &= u_p \qquad (s+1 \le p \le s+t)\\ x_{s+t+p} &= \prod_{i=1}^{s+t} u_i^{-a_{ip}} z_p \quad (1 \le p \le l). \end{aligned}$$

Then $\tilde{f} = \tilde{f}(\alpha; x)$ satisfies

$$(\theta_p + \sum_{j=1}^{l} a_{pj}\theta_{s+t+j} + \alpha_p)\tilde{f} = 0$$

$$(p = 1, \ldots s)$$

$$(\theta_p - \sum_{j=1}^{l} a_{pj}\theta_{s+t+j} - (\alpha_p - 1))\tilde{f} = 0$$

$$(p = s+1, \ldots s+t)$$

$$(\partial^u - \partial^v)\tilde{f} = 0$$

$$(u, v \in N^{s+t+l},\ \tilde{A}u = \tilde{A}v),$$

where

$$\tilde{A} = (1_1, \ldots, 1_s, -1_{s+1}, \ldots, -1_{s+t}, A) \in \mathrm{Mat}_{s+t,s+t+l}(Z)$$

This is the $\tilde{A}$-hypergeometric system with parameter $(-\alpha_1, \ldots, -\alpha_s, \alpha_{s+1}-1, \ldots, \alpha_{s+t}-1)$.

f: non-confluent

$\leftrightarrow$

For each variable z_j, the degrees of m_j in the numerators and in the denominators are equal.

$\leftrightarrow$

$\sum_{i=1}^{s} a_{ij} = \sum_{i=s+1}^{s+t} a_{ij} + 1$
for all $j = 1, \ldots, l$.

Gauss's case: $s = 2$, $t = 1$, $l = 1$, $A = {}^t(1, 1, 1)$.

• Definition of A-Hypergeometric Systems.

$$\begin{aligned} A :&= (a_1, a_2, \ldots, a_n) = (a_{ij}) \\ &\in \mathrm{Mat}_{d\times n}(Z). \end{aligned}$$

Assume $\mathrm{rank}(A) = d$.

$$I_A := \langle \partial^u - \partial^v \mid Au = Av \quad (u, v \in N^n) \rangle.$$

$$D := C\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle.$$

$$\begin{aligned} &H_A(\alpha) \\ &= D/(DI_A + \sum_{i=1}^{d} D(\sum_{j=1}^{n} a_{ij}\theta_j - \alpha_i)) \end{aligned}$$

is the A-hypergeometric system with parameter $\alpha = {}^t(\alpha_1, \ldots, \alpha_d) \in C^d$.

- **Holonomicity and Regularity**

Theorem. [GKZ], [Adolphson], [GGP]
$H_A(\alpha)$ is holonomic.

A is said to be homogeneous if all a_j lie on one hyperplane off the origin.

Theorem. [Hotta]
If A is homogeneous, then $H_A(\alpha)$ is regular holonomic.

Theorem. [SST]
If A is not homogeneous, then $H_A(\alpha)$ is not regular for generic α.

Trios.

[GKZ]: Gelfand, Kapranov, Zelevinsky.

[GGP]: Gelfand, Graev, Postnikov.

[SST]: Saito, Sturmfels, Takayama.

July 12, 05

• Indicial systems

How to find exponents is given.

- **Weight.**

$D := C\langle x, \partial\rangle$, the n-th Weyl algebra.

Let $w = (w_1, \ldots, w_n) \in R^n$.

Consider a weight vector $(-w, w)$:

$$\begin{aligned} \text{weight of } x_j &= -w_j, \\ \text{weight of } \partial_j &= w_j. \end{aligned}$$

Then $\mathrm{gr}_{(-w,w)} D = D$,

for $\mathrm{in}_{(-w,w)}(x_j \partial_j) = \mathrm{in}_{(-w,w)}(1)$,

hence $\partial_j x_j = x_j \partial_j + 1$ in $\mathrm{gr}_{(-w,w)} D$.

- **Initial ideals.**

For a left ideal J of D, define

$$\mathrm{in}_{(-w,w)}J := D\cdot\{\mathrm{in}_{(-w,w)}P : P \in J\}.$$

Here

$$\mathrm{in}_{(-w,w)}P := \sum_{w\cdot(\beta-\alpha)=\max.} a_{\alpha\,\beta}x^{\alpha}\partial^{\beta},$$

when $P = \sum a_{\alpha\,\beta}x^{\alpha}\partial^{\beta}$.

Then

$\mathrm{in}_{(-w,w)}J$ is a left ideal of $D = \mathrm{gr}_{(-w,w)}D$,

called the <u>initial ideal</u> of J.

Example. $(n = 1,\ D = \langle z, \frac{d}{dz} \rangle.)$

$$P = (\theta + \gamma)\frac{d}{dz} - (\theta + \alpha)(\theta + \beta).$$

$$J = DP.$$

$$w = 1. \quad (z \cdots -1,\ \frac{d}{dz} \cdots 1.)$$

Then

$$\mathrm{in}_{(-1,1)} P = (\theta + \gamma)\frac{d}{dz},$$

$$\mathrm{in}_{(-1,1)} J = D(\theta + \gamma)\frac{d}{dz}.$$

- **Indicial ideals.**

$R := C(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle.$

$C[\theta] := C[\theta_1, \ldots, \theta_n].$

For a left ideal I of D, define

$\tilde{I} := RI \cap C[\theta]$: the distraction of I.

For a left ideal J of D, define

$$\begin{aligned} \mathrm{ind}_w J : &= \widetilde{\mathrm{in}_{(-w,w)} J} \\ &= (R\, \mathrm{in}_{(-w,w)} J) \cap C[\theta], \end{aligned}$$

called the indicial ideal of J.

Call a root of $\mathrm{ind}_w J$ an exponent w.r.t. w.

Example. $(n = 1,\ D = \langle z, \frac{d}{dz} \rangle.)$

$$J = D((\theta + \gamma)\tfrac{d}{dz} - (\theta + \alpha)(\theta + \beta)).$$

$$\mathrm{in}_{(-1,1)} J = D(\theta + \gamma)\tfrac{d}{dz}.$$

$$\begin{aligned} R\,\mathrm{in}_{(-1,1)} J &= R(\theta + \gamma)\frac{d}{dz} \\ &= R(\theta + \gamma - 1)\theta. \end{aligned}$$

$$\begin{aligned} \mathrm{ind}_1 J &= \mathrm{in}_{\widetilde{(-1,1)}} J \\ &= \langle (\theta + \gamma - 1)\theta \rangle \subseteq C[\theta]. \end{aligned}$$

Exponents of J w.r.t. 1
$= \text{Roots of } (\theta + \gamma - 1)\theta = 0$
$= \{0, -\gamma + 1\}.$

Assume A to be homogeneous (today and tomorrow).

Theorem 2-1.

$\alpha \in C^d$: any, fixed.

$w \in R^n$: generic.

nec. when taking an intersection with $C[\theta]$

Then

$$\mathrm{rank}(H_A(\alpha)) = \mathrm{rank}(\mathrm{ind}_w H_A(\alpha)).$$

$H_A(\alpha) = DI_A + D\langle A\theta - \alpha\rangle$
$D\langle A\theta - \alpha\rangle = \sum_{i=1}^{d} D(\sum_{j=1}^{n} a_{ij}\theta_j - \alpha_i)$.
rank($H_A(\alpha)$) is a dim. of solution space at a generic point.

$\mathrm{rank}(\mathrm{ind}_w H_A(\alpha)) := \dim_C C[\theta]/\mathrm{ind}_w H_A(\alpha)$.

So we want to find exponents.

- **Fake indicial ideals.**

Clearly,

$$D\,\mathrm{in}_w I_A + D\langle A\theta - \alpha\rangle \subseteq \mathrm{in}_{(-w,w)} H_A(\alpha).$$

Hence define an ideal of $C[\theta]$

$$\begin{aligned}\mathrm{find}_w H_A(\alpha) :&= \widetilde{D\,\mathrm{in}_w I}_A + \langle A\theta - \alpha\rangle \\ &\subseteq \mathrm{ind}_w H_A(\alpha),\end{aligned}$$

called the <u>fake indicial ideal</u> of $H_A(\alpha)$. (If α is generic, the equality holds.)

Its roots are called <u>fake exponents</u>.

Try to find fake exponents instead of exponents. To do this, we want to know $\widetilde{D\,\mathrm{in}_w I}_A$.

- **Standard pairs.**

M: a monomial ideal of
$$C[\partial] := C[\partial_1, \dots, \partial_n].$$
(e.g. $M = \mathrm{in}_w I_A$, w: generic.)

Identify M with

$\{a \in N^n : \partial^a \in M\} =: M.$

For $a \in N^n$, $\sigma \subseteq \{1, 2, \dots, n\}$,

(a, σ) is a <u>standard pair</u> of M if

(1) $a_j = 0 \quad (\forall j \in \sigma)$,

(2) $(a + N^\sigma) \cap M = \emptyset$,

(3) $(a + N^{\sigma \cup \{l\}}) \cap M \neq \emptyset \quad (\forall l \notin \sigma)$,

where
$N^\sigma = \{u \in N^n : u_j = 0 \, (\forall j \notin \sigma)\}.$

$$\mathcal{S}(M) := \{\text{standard pairs of } M\}.$$

Then

$$N^n \setminus M = \bigcup_{(a,\sigma)\in\mathcal{S}(M)} (a + N^\sigma).$$

Example.

$$M = \langle \partial_1^2\partial_2, \partial_1\partial_2^2 \rangle \subseteq C[\partial_1, \partial_2].$$

Then

$$\begin{aligned}\mathcal{S}(M) &= \{(0,\{1\}),(0,\{2\}),((1,1),\emptyset)\} \\ &= \{(0,*),(*,0),(1,1)\}.\end{aligned}$$

(Put a_j at the j-th place if $j \notin \sigma$,

and $*$ if $j \in \sigma$.)

By considering the Hilbert function, easy to see

$$\dim_{C[\partial]} C[\partial]/M \\ = \max\{\sharp\sigma : (a,\sigma) \in \mathcal{S}(M)\}$$

and

$$\deg(C[\partial]/M) = \sharp\mathcal{T}(M),$$

where

$$\mathcal{T}(M) \\ := \{(a,\sigma) \in \mathcal{S}(M) : \sharp\sigma = \dim C[\partial]/M\}.$$

Theorem 2-2.

Let M be a monomial ideal of $C[\partial]$. Then

$$\widetilde{DM} = \bigcap_{(a,\sigma)\in\mathcal{S}(M)} \langle \theta_j - a_j : j \notin \sigma \rangle.$$

Example. $n = 1, M = \langle \partial^m \rangle$. **Then**

$$\mathcal{S}(M) = \{(0,\emptyset), \ldots , (m-1,\emptyset)\}.$$

$RM = Rx^m\partial^m$, **and**

$$\begin{aligned}\widetilde{DM} &= \langle \theta(\theta-1)\cdots(\theta-m+1)\rangle \\ &= \langle\theta\rangle \cap \langle\theta-1\rangle \cap \\ &\qquad \cdots \cap \langle\theta-(m-1)\rangle.\end{aligned}$$

Back to our case.

For $M = \mathrm{in}_w I_A$ (w: generic),

$$\begin{aligned}&\max\{\sharp\sigma : (a,\sigma) \in \mathcal{S}(\mathrm{in}_w I_A)\}\\ &\qquad = \dim C[\partial]/\mathrm{in}_w I_A\\ &\qquad = \dim C[\partial]/I_A = d.\end{aligned}$$

$$\begin{aligned}&\sharp\mathcal{T}(\mathrm{in}_w I_A)\\ &\qquad = \deg C[\partial]/\mathrm{in}_w I_A\\ &\qquad = \deg C[\partial]/I_A = \mathrm{vol}(A).\end{aligned}$$

$$(\mathrm{vol}(\langle 0, e_1, \ldots, e_d\rangle) = 1.)$$

Theorem 2-3.

Let w be generic. Then

$$\mathrm{find}_w H_A(\alpha) = \bigcap_{(a,\sigma)\in\mathcal{S}(\mathrm{in}_w I_A)} \langle \theta_j - a_j : j \notin \sigma\rangle + \langle A\theta - \alpha\rangle.$$

Corollary 2-4.

Let w be generic. Then

(1)

$$\{\text{fake exponents of } H_A(\alpha) \text{ w.r.t. } (-w,w)\} = \bigcup_{(a,\sigma)\in\mathcal{S}(\mathrm{in}_w I_A)} \{v\in C^n : Av=\alpha, v_j=a_j(\forall j\notin\sigma)\}$$

(2) If α is generic, then

$$\{\text{exponents of } H_A(\alpha) \text{ w.r.t. } (-w,w)\} = \{\text{fake exponents of } H_A(\alpha) \text{ w.r.t. } (-w,w)\} = \bigcup_{(a,\sigma)\in\mathcal{T}(\mathrm{in}_w I_A)} \{v\in C^n : Av=\alpha, v_j=a_j(\forall j\notin\sigma)\}$$

This proves

$\mathrm{rank}\, H_A(\alpha) = \sharp\mathcal{T}(\mathrm{in}_w I_A) = \mathrm{vol}(A)$,

when α is generic.

Example.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = [a_1, a_2, a_3, a_4].$$

Then $I_A = \langle \partial_1\partial_2 - \partial_3\partial_4 \rangle$.

$w := (0,0,0,1)$. Then $\mathrm{in}_w I_A = \langle \partial_3\partial_4 \rangle$,

$\mathcal{S}(\mathrm{in}_w I_A) = \{(*,*,0,*),(*,*,*,0)\}$.

(1) v: fake exp. of type $(*,*,0,*)$.

Then $v_1a_1 + v_2a_2 + v_4a_4 = \alpha$.

Hence $v = (\alpha_1 - \alpha_3, \alpha_2 - \alpha_3, 0, \alpha_3)$.

(2) v: fake exp. of type $(*,*,*,0)$.

Then $v_1a_1 + v_2a_2 + v_3a_3 = \alpha$.

Hence $v = (\alpha_1, \alpha_2, -\alpha_3, 0)$.

July 13, 05

• A-hypergeometric series

- **Notation.**

For $v \in C^n$, $u \in N^n$, define $[v]_u \in C$ by

$$\partial^u(x^v) = [v]_u x^{v-u},$$

or $[v]_u = \prod_{u_i>0} \prod_{m=0}^{u_i-1}(v_i - m)$.

For $u \in Z^n$, define $u_+, u_- \in N^n$ by

$$(u_+)_i := \begin{cases} u_i & (u_i \geq 0) \\ 0 & (u_i \leq 0), \end{cases}$$

$$(u_-)_i := \begin{cases} 0 & (u_i \geq 0) \\ -u_i & (u_i \leq 0). \end{cases}$$

$$L := \mathrm{Ker}_Z(A) = \{u \in Z^n : Au = 0\}.$$

Lemma 3-1.

Let $v \in (C \setminus Z_{<0})^n$. Then

$$\phi_v := \sum_{u \in L} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u}$$

is a formal sol. of $H_A(Av)$.

$(v_i \notin Z_{<0}\ (\forall i) \Rightarrow [v+u]_{u_+} \neq 0.)$

Proposition 4-2.

Let w be generic.

Let $v \in (C \setminus Z_{<0})^n$ be a fake exp. of $H_A(Av)$ w.r.t. w.

Then ϕ_v is a solution,

$\phi_v = x^v(1 + w\text{-higher terms})$,

(i.e., $[v]_{u_-} \neq 0, w \cdot u \leq 0 \Rightarrow u = 0.$),

so v is an exponent.

Since $\mathrm{find}_w H_A(Av)$ does not change if we replace w by w' with $\mathrm{in}_{w'} I_A = \mathrm{in}_w I_A$, see

$$\phi_v \in x^v \cdot C[[C(w)^*_Z]],$$

where

$$C(w)^*_Z = \left\{ u \in Z^n : \begin{array}{c} \mathrm{in}_{w'} I_A = \mathrm{in}_w I_A \\ \Rightarrow u \cdot w' \geq 0 \end{array} \right\}.$$

Let $l_1, \ldots, l_n \in C(w)$ be a basis of Z^n, where

$$C(w) = \{w' : \mathrm{in}_{w'} I_A = \mathrm{in}_w I_A\}.$$

Then

$$C(w)^*_Z \subseteq N l^*_1 \bigoplus \cdots \bigoplus N l^*_n.$$

($l^*_1, \ldots, l^*_n$ is the dual basis.)

So $\phi_v \in x^v \cdot C[[x^{l^*_1}, \ldots, x^{l^*_n}]]$,

and the series converges if

$0 < |x^{l^*_j}| \ll 1$ for all j.

- **Generic case.**

Let w, α be generic so that

•{exp's of $H_A(\alpha)$ w.r.t. w}
= {fake exp's of $H_A(\alpha)$ w.r.t. w}
= $\{\alpha^{(a,\sigma)} : (a,\sigma) \in \mathcal{T}(\mathrm{in}_w I_A)\}$.
•All $\alpha^{(a,\sigma)}$ are distinct.
•Any comp. of $\alpha^{(a,\sigma)} \notin Z_{<0}$,

where $v = \alpha^{(a,\sigma)}$ is the fake exponent with $v_j = a_j$ $(j \notin \sigma)$, $Av = \alpha$.

Proposition 3-3.

$$\{\phi_v : v = \alpha^{(a,\sigma)},\ (a,\sigma) \in \mathcal{T}(\mathrm{in}_w I_A)\}$$

is a basis of a sol. space of $H_A(\alpha)$.

(rank $H_A(\alpha) = \sharp\mathcal{T}(\mathrm{in}_w I_A) = \deg I_A =$ vol(A).)

- **Negative support.**

Want to consider ϕ_v when v has a negative integer component.

For $v \in C^n$, put

$$\mathrm{nsupp}(v) := \{i \in \{1, 2, \ldots, n\} : v_i \in Z_{<0}\},$$

called the <u>negative support</u> of v.

$$N_v := \{u \in L : \mathrm{nsupp}(v + u) = \mathrm{nsupp}(v)\}.$$

Lemma 3-4.

$$u \in N_v \Rightarrow [v + u]_{u_+} \neq 0.$$

- **Log-free series.**

Redefine

$$\phi_v := \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u}.$$

(Two def's of ϕ_v for $v \in (C \setminus Z_{<0})^n$ coincide.)

$v \in C^n$ has <u>minimal negative support</u>

if nsupp(v) is minimal

among nsupp$(v+u)$ $(u \in L)$.

Proposition 3-5.

ϕ_v is a formal sol. of $H_A(Av)$

$\Leftrightarrow$ v has min. neg. support.

(Pf. of $\Rightarrow$) Contraposition.

Suppose $\exists u \in L$ s.t.

$$\mathrm{nsupp}(v+u) \subsetneq \mathrm{nsupp}(v).$$

Then $\partial^{u_-}(x^v) \neq 0$, for

otherwise $\exists i$ s.t.

$u_i < 0, v_i \in N, v_i + u_i \in Z_{<0}$,

i.e., $i \in \mathrm{nsupp}(v+u) \setminus \mathrm{nsupp}(v)$,

contradicting the assumption.

ϕ_v doesn't have a term of x^{v+u} by definition.

Hence the coef. of

$$x^{v-u_-} = x^{(v+u)-u_+}.$$

in $(\partial^{u_+} - \partial^{u_-})\phi_v$ is not zero.

Hence ϕ_v isn't a formal sol.

Theorem 3-6.

Let w be generic.

Let v be a fake exp. w.r.t. w of $H_A(Av)$ with min. neg. support.

Then

ϕ_v is a sol. (v is an exp.)

ϕ_v doesn't contain a term with any other fake exp.

Proof. Suppose $v+u$ is a fake exp. s.t. x^{v+u} appears in ϕ_v, and $w \cdot u > 0$.

Then $\partial^{u_+}(x^{v+u}) = 0$, for

$\partial^{u_+} \in \mathrm{in}_w I_A$, and $v+u$ is a fake exp.

So $\exists i$ s.t.

$u_i > 0$, $v_i + u_i \in N$, $v_i \in Z_{<0}$,

contradicting $u \in N_v$.

Theorem 3-7.

$$\left\{ \phi_v : \begin{array}{c} v \text{ is a fake exp. w.r.t. } w \\ \text{of } H_A(\alpha) \text{ with} \\ \text{min. neg. support} \end{array} \right\}$$

is a basis of

$$\left\{ \begin{array}{l} \text{log-free series sol's of } H_A(\alpha) \\ \text{in the direction of } w \end{array} \right\}.$$

Proof.

Let ϕ be a log-free series sol. of $H_A(\alpha)$ s.t. $\phi = x^v(1 + (w\text{-higher terms}))$.

Then v is a fake exp.

Then, as before, v has to have min. neg. supp.

Consider $\phi - \phi_v$, and repeat the process to get an expression of ϕ as a linear combination of ϕ_v's.

Example.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = [a_1, a_2, a_3, a_4].$$

Then $I_A = \langle \partial_1 \partial_2 - \partial_3 \partial_4 \rangle$.

$w := (0,0,0,1)$. Then $\mathrm{in}_w I_A = \langle \partial_3 \partial_4 \rangle$,

$\mathcal{S}(\mathrm{in}_w I_A) = \{(*,*,0,*),(*,*,*,0)\}$.

Fake exp's are

$$e^{(1)} := (\alpha_1 - \alpha_3, \alpha_2 - \alpha_3, 0, \alpha_3),$$
$$e^{(2)} := (\alpha_1, \alpha_2, -\alpha_3, 0).$$

$$L = Z(-1,-1,1,1).$$

Let $u := (-1,-1,1,1)$, and $l \in Z$.

$$\mathrm{nsupp}(e^{(i)}) \subseteq \mathrm{nsupp}(e^{(i)} + lu)$$
$$\Rightarrow l \in N.$$

(Look at the 3rd or 4th comp.)

- **Generic case.**

$(\alpha_i, \alpha_i - \alpha_j \notin Z (\forall i \neq j).)$

Then

$$\mathrm{nsupp}(e^{(i)}) = \mathrm{nsupp}(e^{(i)} + lu) \\ \Leftrightarrow l \in N.$$

$$\phi_{e^{(1)}} \\ = x^{e^{(1)}} \sum_{l=0}^{\infty} \frac{[e^{(1)}]_{(l,l,0,0)}}{[e^{(1)} + lu]_{(0,0,l,l)}} \left(\frac{x_3 x_4}{x_1 x_2} \right)^l$$

Here

$$\begin{aligned} &[e^{(1)}]_{(l,l,0,0)} \\ &= (\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_3 - l + 1) \\ &\quad \times (\alpha_2 - \alpha_3) \cdots (\alpha_2 - \alpha_3 - l + 1) \\ &= (\alpha_3 - \alpha_1) \cdots (\alpha_3 - \alpha_1 + l - 1) \\ &\quad \times (\alpha_3 - \alpha_2) \cdots (\alpha_3 - \alpha_2 + l - 1) \\ &= (\alpha_3 - \alpha_1)_l (\alpha_3 - \alpha_2)_l. \end{aligned}$$

Similarly

$$[e^{(1)} + lu]_{(0,0,l,l)} = (\alpha_3 + 1)_l l!.$$

Hence

$$\phi_{e^{(1)}}$$

$$= x_1^{\alpha_1 - \alpha_3} x_2^{\alpha_2 - \alpha_3} x_4^{\alpha_3}$$

$$\times F(\alpha_3 - \alpha_1, \alpha_3 - \alpha_2, \alpha_3 + 1; \frac{x_3 x_4}{x_1 x_2})$$

$$= x_1^{-\alpha+\gamma-1} x_2^{-\beta+\gamma-1} x_4^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma; \frac{x_3 x_4}{x_1 x_2})$$

Similarly

$$\phi_{e^{(2)}}$$

$$= x_1^{\alpha_1} x_2^{\alpha_2} x_3^{-\alpha_3}$$

$$\times F(-\alpha_1, -\alpha_2, -\alpha_3 + 1; \frac{x_3 x_4}{x_1 x_2}).$$

$$= x_1^{-\alpha} x_2^{-\beta} x_3^{\gamma-1} F(\alpha, \beta, \gamma; \frac{x_3 x_4}{x_1 x_2})$$

when $\alpha_1 = -\alpha, \alpha_2 = -\beta, \alpha_3 = 1 - \gamma$.

Set $x_1 = x_2 = x_3 = 1$ to get classical sol's.

- Case $\alpha_1 = 1, \alpha_2 = \alpha_3 = 2$.

Then

$$\begin{aligned} e^{(1)} &= (-1, 0, 0, 2) \\ e^{(2)} &= (1, 2, -2, 0). \end{aligned}$$

$\alpha \notin NA$. ($\alpha \notin R_{\geq 0}A$.) implies

$e^{(1)}, e^{(2)}$ have min. neg. supports.

$$\begin{aligned} \mathrm{nsupp}(e^{(1)} + lu) &= \mathrm{nsupp}(e^{(1)}) = \{1\} \\ &\Leftrightarrow l = 0. \end{aligned}$$

$$\begin{aligned} \mathrm{nsupp}(e^{(2)} + lu) &= \mathrm{nsupp}(e^{(2)}) = \{3\} \\ &\Leftrightarrow l = 0, 1. \end{aligned}$$

$$\begin{aligned} \phi_{e^{(1)}} &= x^{e^{(1)}} = x_1^{-1} x_4^2 \\ &= x_1^{-1} x_4^2 \, F(1, 0, 3, \frac{x_3 x_4}{x_1 x_2}). \end{aligned}$$

$$\begin{aligned} \phi_{e^{(2)}} &= x^{e^{(2)}} (1 + \frac{[e^{(2)}]_{(1,1,0,0}}{[e^{(2)} + u]_{(0,0,1,1)}} \frac{x_3 x_4}{x_1 x_2}) \\ &= x_1 x_2^2 x_3^{-2} (1 - 2 \frac{x_3 x_4}{x_1 x_2}) \\ &= x_1 x_2^2 x_3^{-2} - 2 x_2 x_3^{-1} x_4. \end{aligned}$$

Note that

$F(-1, -2, -1; \frac{x_3 x_4}{x_1 x_2})$ is not defined.

The 2nd and 3rd talks are taken from

Saito, Sturmfels, Takayama: Gröbner deformations of hypergeometric differential equations, Springer 2000.

July 14, 05

A-equivalence

Contents.

(1) Classifying A-hypergeometric systems $H_A(\alpha)$ w.r.t. D-isomorphisms using finite sets $E_\tau(\alpha)$.

(2) Examples: Getting used to $E_\tau(\alpha)$.

(3) Conditions in terms of $E_\tau(\alpha)$.

The main reference is

Saito, Compositio Math. 128 (2001) 323–338.

• A-hypergeometric systems

$$A := (a_1, \dots, a_n) = (a_{ij}) \in M_{d\times n}(\mathrm{Z})$$

Assume rank$(A) = d$.

$$I_A := \langle \partial^u - \partial^v \,|\, Au = Av,\ u, v \in \mathrm{N}^n \rangle$$
$$\subseteq C[\partial] = C[\partial_1, \dots, \partial_n].$$
$$D = C\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$$

For $\alpha = {}^t(\alpha_1, \dots, \alpha_d) \in C^d$,

$$H_A(\alpha) := D/(D\cdot I_A + \sum_{i=1}^{d} D(\sum_{j=1}^{n} a_{ij}\theta_j - \alpha_i));$$

the A-hypergeometric system
with parameter α. $\qquad (\theta_j = x_j\partial_j)$

Problem

Give conditions on A, α, α'
for $H_A(\alpha) \simeq H_A(\alpha')$.

• Finite Sets $E_\tau(\alpha)$

$A := \{ a_1, \dots , a_n \}$.

$\mathrm{R}_{\geq 0}A = \sum_{j=1}^{n} \mathrm{R}_{\geq 0}a_j$.

Assume $\mathrm{R}_{\geq 0}A$ to be strongly convex for simplicity.

τ: a face of $\mathrm{R}_{\geq 0}A$

For $\alpha \in C^d$, define a (finite) set:

$$E_\tau(\alpha) := \{ \lambda \in C(A \cap \tau)/\mathrm{Z}(A \cap \tau) \\ : \ \alpha - \lambda \in \mathrm{N}A + \mathrm{Z}(A \cap \tau) \}.$$

Here

- $\mathrm{N} = \{ 0, 1, 2, \dots \}$
- $C(A \cap \tau) = \mathrm{Z}(A \cap \tau) = \{ 0 \}$ when $\tau = \{ 0 \}$.

Example. $A = (1)$.

Then $\mathrm{R}_{\geq 0}A = \mathrm{R}_{\geq 0}$ has only two faces: $\mathrm{R}_{\geq 0}$, $\{0\}$.

- $E_{\mathrm{R}_{\geq 0}A}(\alpha) = \{\, \alpha \bmod \mathrm{Z} \,\}$.
- $E_{\{0\}}(\alpha) = \{0\}$ or $\emptyset$.
- $E_{\{0\}}(\alpha) = \{0\} \Leftrightarrow \alpha \in \mathrm{N}$.

Remark.

$R_{\tau,l} := x^l \cdot C[[x^m : m \in \mathrm{N}^{\tau^c} \times \mathrm{Z}^{\tau}]]$
is a D-module $\quad (l \in C^{\tau})$,

where

$C^{\tau} := \{\, u \in C^n \,:\, u_j = 0\,(a_j \notin \tau)\,\}$,

$N^{\tau^c} := \{\, u \in N^n \,:\, u_j = 0\,(a_j \in \tau)\,\}$.

Proposition 4-1. $Al \in E_\tau(\alpha)$

$\Leftrightarrow \mathrm{Hom}_D(H_A(\alpha), R_{\tau,l}) \neq 0$.

Proof. $(\Leftarrow)$ Let

$0 \neq \phi \in \mathrm{Hom}_D(H_A(\alpha), R_{\tau,l})$.

Then $0 \neq \phi(\bar{1}) \in R_{\tau,l}$ has weight α.

So $\exists m \in N^{\tau^c} \times Z^\tau$ s.t. $A(l+m) = \alpha$.
Hence $Al \in E_\tau(\alpha)$.

$(\Rightarrow)$ Suppose $Al \in E_\tau(\alpha)$. Then

$\exists m \in N^{\tau^c} \times Z^\tau$ s.t. $A(l+m) = \alpha$.

Take $v \in L = \mathrm{Ker}_Z(A)$ s.t.
$l+m+v$ has min. neg. supp. with
$\mathrm{nsupp}(l+m+v) \subseteq \mathrm{nsupp}(l+m)$

See $m+v \in N^{\tau^c} \times Z^\tau$.

So we may assume $l+m$ has min. neg. supp. Then

$0 \neq \phi_{l+m} \in R_{\tau,l}$ is a formal sol.

• Classification

For $\alpha, \alpha' \in C^d$, define

$$\alpha \preceq \alpha' \quad \overset{\text{def.}}{\Leftrightarrow} \quad E_\tau(\alpha) \subseteq E_\tau(\alpha') \quad (\forall \tau).$$

$$\alpha \sim \alpha' \quad \overset{\text{def.}}{\Leftrightarrow} \quad E_\tau(\alpha) = E_\tau(\alpha') \quad (\forall \tau).$$

The previous proposition implies
$H_A(\alpha) \simeq H_A(\alpha') \Rightarrow \alpha \sim \alpha'$.
Indeed we have

Theorem 4-2.

$$H_A(\alpha) \simeq H_A(\alpha') \Leftrightarrow \alpha \sim \alpha'.$$

In principle, all D-invariants
can be expressed in terms of $E_\tau(\alpha)$.

• Some properties of $E_\tau(\alpha)$

Proposition 4-3.

(1) $$E_{\mathbb{R}_{\geq 0}A}(\alpha) = \{\,\alpha \mod \mathbb{Z}A\}.$$

$$\begin{aligned} E_{\mathbb{R}_{\geq 0}A}(\alpha) &= E_{\mathbb{R}_{\geq 0}A}(\alpha') \\ &\Leftrightarrow \alpha - \alpha' \in \mathbb{Z}A. \end{aligned}$$

(2)

- $E_{\{0\}}(\alpha) = \{\,0\,\}$ or $\emptyset$.
- $E_{\{0\}}(\alpha) = \{\,0\,\} \Leftrightarrow \alpha \in \mathbb{N}A$.

(3) For faces $\tau \preceq \sigma$,
$\exists$ a natural map: $E_\tau(\alpha) \to E_\sigma(\alpha)$.
In particular,

$$E_\tau(\alpha) \neq \emptyset \Rightarrow E_\sigma(\alpha) \neq \emptyset.$$

(4)

$$\begin{aligned} &|E_\tau(\alpha)| \\ &\quad \leq [(\mathbb{Q}(A \cap \tau)) \cap \mathbb{Z}A : \mathbb{Z}(A \cap \tau)]. \end{aligned}$$

(5) For a facet σ,

$$E_{\sigma}(\alpha) \neq \emptyset \Leftrightarrow F_{\sigma}(\alpha) \in F_{\sigma}(NA).$$

Here $F_{\sigma} : C^{d} \longrightarrow C$ is the linear map determined uniquely by the following conditions:

(i) $F_{\sigma}(\sigma) = 0$,

(ii) $F_{\sigma}(ZA) = Z$,

(iii) $F_{\sigma}(R_{\geq 0}A) \geq 0$.

• In terms of E_τ

$$ZA = \{a \,:\, E_{R_{\geq 0}A}(a) = \{0\}\}.$$

$$\begin{aligned} NA &= \{a \,:\, 0 \in E_\tau(a) \quad (\forall \tau)\} \\ &= \{a \,:\, 0 \in E_{\{0\}}(a)\}. \end{aligned}$$

$$S := \{a \in ZA \,:\, E_\sigma(a) \neq \emptyset \\ (\forall \sigma : \text{facets})\}.$$

$$S_2 := \{a \in ZA \,:\, 0 \in E_\sigma(a) \\ (\forall \sigma : \text{facets})\}.$$

Then $NA \subseteq S_2 \subseteq S \subseteq R_{\geq 0}A \cap ZA$.

NA : scored $\Leftrightarrow NA = S$.

$C[NA]$ satisfies $(S_2) \Leftrightarrow NA = S_2$.

$\boldsymbol{R_{\geq 0} A \cap Z A}$ **is not, in general, a union of** $\boldsymbol{A}$**-equivalence classes.**

Probably the simplicity is more important than normality in the theory.

Simplicity: All $\boldsymbol{A}$**-equivalence classes are Zariski-dense.**

Example.

$$\boldsymbol{A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}.}$$

Then $\boldsymbol{NA}$ **is not normal, but** $\boldsymbol{A}$ **satisfies the simplicity condition.**
$\boldsymbol{R_{\geq 0} A \cap Z A}$ **is not a union of** $\boldsymbol{A}$**-equivalence classes.**

The condition $\boldsymbol{C[NA]}$ **is Cohen-Macaulay is also described by** $\boldsymbol{E_{\tau}}$ **as shown in**

Matusevich, Miller, Combinatorics of rank jumps in simplicial hypergeometric systems.

- α is nonresonant.

$\overset{\text{def.}}{\Leftrightarrow}$ $F_\sigma(\alpha) \notin Z \quad (\forall\sigma\text{: facet}).$

$\Leftrightarrow$ $\alpha + ZA$ is one equivalence class.

- α is seminonresonant.

$\overset{\text{def.}}{\Leftrightarrow}$ $F_\sigma(\alpha) \notin F_\sigma(NA) \quad (\forall\sigma\text{: facet}).$

$\Leftrightarrow$ $E_\tau(\alpha) = \emptyset \quad (\forall\tau\text{: proper faces}).$

$\Leftrightarrow$ $[\alpha]$ is the smallest equiv. class in $\alpha + ZA$.

Remark. The above definition of seminonresonance is different from the usual one. The usual definition requires $F_\sigma(\alpha) \notin N$ instead of $F_\sigma(\alpha) \notin F_\sigma(NA)$. But this notion is not invariant under A-equivalence.

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} = (a_1, a_2, a_3, a_4).$$

$$\{\text{ faces }\} = \{\, \mathrm{R}_{\geq 0}A, \sigma_1, \sigma_4, \{\, 0 \,\}\,\}$$

$\sigma_1 := R_{\geq 0}a_1, \quad \sigma_4 := R_{\geq 0}a_4.$

$|E_\tau(\alpha)| = 0$ or 1 ($\forall\tau$: faces).

Let $\alpha \in \mathrm{Z}A$. ZA has 5 classes:

1. $E_{\mathrm{R}_{\geq 0}A}(\alpha) = \{0\}$,
   $E_{\sigma_1}(\alpha) = E_{\sigma_4}(\alpha) = E_{\{0\}}(\alpha) = \emptyset$.
2. $E_{\mathrm{R}_{\geq 0}A}(\alpha) = E_{\sigma_1}(\alpha) = \{0\}$.
   $E_{\sigma_4}(\alpha) = E_{\{0\}}(\alpha) = \emptyset$.
3. $E_{\mathrm{R}_{\geq 0}A}(\alpha) = E_{\sigma_4}(\alpha) = \{0\}$.
   $E_{\sigma_1}(\alpha) = E_{\{0\}}(\alpha) = \emptyset$.
4. $E_{\mathrm{R}_{\geq 0}A}(\alpha) = E_{\sigma_1}(\alpha) = E_{\sigma_4}(\alpha) = \{0\}$. $E_{\{0\}}(\alpha) = \emptyset$.
   i.e., $\alpha = {}^t(1, 2)$.
5. $E_{\mathrm{R}_{\geq 0}A}(\alpha) = E_{\sigma_1}(\alpha) = E_{\sigma_4}(\alpha) = E_{\{0\}}(\alpha) = \{0\}$, i.e. $\alpha \in \mathrm{N}A$.